Int J Theor Phys (2010) 49: 1195–1199
DOI 10.1007/s10773-010-0299-0

# Non-commutative Geometry in Massless and Massive Particles

**M.R. Pahlavani · B. Pourhassan**

Received: 3 January 2010 / Accepted: 25 February 2010 / Published online: 19 March 2010


**Abstract** In this paper, we study the symmetries of massless and massive particles action. By considering the non-commutative space-time, we find appropriate non-commutative relation for relativistic particles which leaves invariant the non-commutative Minkowski space-time. We show that non-commutativity break the scale and conformal invariance in massless and massive action. So, in non-commutative space-time the massless and massive particles have same symmetry.

**Keywords** Twisted · Poincaré and conformal transformation · Non-commutative geometry

## 1 Introduction

Relativistic particle theory may be seems as a first version of string theory and general relativity, also the connection between string theory and non-commutative geometry has been known. This is because it is believed that in quantum theories containing gravity, space-time must change its nature at distances comparable to the Planck scale. Quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length [1], non-vanishing commutation relation between the space-time coordinates is the model for these effects. Many developments in string and superstring theories were reached in this direction [2–4], and it is now believed that non-commutative geometry naturally arises from the string theory.

As we know [5] found that the massless particle action has a larger set of space-time invariance and then the appearance of non-vanishing mass may be associated the breaking of some of these extra invariance such as conformal invariance. In commutative space the

M.R. Pahlavani
Department of Physics, Islamic Azad University—Ayatollah Amoli Branch, P.O. Box 678, Amol, Iran
e-mail: m.pahlavani@umz.ac.ir

B. Pourhassan (✉)
Department of Physics, Mazandaran University, P.O. Box 47416-95447, Babolsar, Iran
e-mail: b.pourhassan@umz.ac.ir

massless particle action had a Poincaré and conformal invariance but the massive just had a Poincaré invariance. So, this give us motivation to investigate the massless and massive action in non-commutative space. Also we show that the massless and massive particles action have a same invariance in non-commutative space, because the conformal invariance broken for the two actions.

In this work we consider relativistic particles action and its symmetries. We apply this symmetries to non-commutative space-time and find appropriate relation which leaves Poincaré invariance. Also we write conformal algebra in terms of new coordinates.

## 2 Relativistic Particles Action and Its Symmetries

First, we review the symmetries of the relativistic particles action. A relativistic particle describes is space-time a one-parameter trajectory $x^\mu(\tau)$. The dynamics of the particle must has reparametrization invariance, it means that the value of the action is independent of the parametrization. A possible form of the action are given by [5],

$$S = \frac{1}{2}\int d\tau(\lambda^{-1}\dot{x}^2 - \lambda m^2), \tag{1}$$

where $\tau$ is an arbitrary parameter along the particle's world-line and a dot denotes derivatives with respect to $\tau$ and also we use the neutral unit ($\hbar = c = 1$). Action (1) defines a generally covariant one-dimensional field theory where the particle mass $m$ plays the role of a cosmological constant [6–8]. We can take $m = 0$ limit in action (1) and go to massless limit which is described by the following action,

$$S = \frac{1}{2}\int d\tau\lambda^{-1}\dot{x}^2, \tag{2}$$

where massless limit is usually considered to describe the high energy limit of relativistic particle theory.

Let us now consider space-time symmetries of action (2). In general, we can write four transformations which leaves action (2) invariant. Indeed action (2) has following symmetries, translation with generators $P_\mu = \partial_\mu$,

$$\begin{aligned} x^{\mu\prime} &\to x^\mu + a^\mu, \\ \lambda' &\to \lambda, \end{aligned} \tag{3}$$

and rotation with generators $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$,

$$\begin{aligned} x^{\mu\prime} &\to x^\mu + \omega^\mu_\nu x^\nu, \\ \lambda' &\to \lambda, \end{aligned} \tag{4}$$

where $\omega$ is antisymmetric tensor. Translation (3) and rotation (4) construct the Poincaré transformation. Both action (1) and (2) are invariant under the Poincaré transformations, but the massless action has a larger symmetries. It is invariant under the scale transformation with generators $D = x^\mu\partial_\mu$,

$$\begin{aligned} x^{\mu\prime} &\to x^\mu + \alpha x^\mu, \\ \lambda' &\to 2\alpha\lambda, \end{aligned} \tag{5}$$

where $\alpha$ is an infinitesimal constant.

Also action (2) is invariant under the conformal transformation with generators $K_\mu = (2x_\mu x^\nu - \delta_\mu^\nu x^2)\partial_\nu$,

$$x^{\mu\prime} \to x^\mu + (2x^\mu x^\nu - \eta^{\mu\nu} x^2) b_\nu, \qquad \lambda' \to 4\lambda x.b, \tag{6}$$

where $b_\mu$ is a constant vector.

Generally we can write total vector field as,

$$V = a^\mu P_\mu - \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \alpha D + b^\mu K_\mu. \tag{7}$$

The generators of the vector field (7) satisfy the conformal space-time algebra,

$$[P_\mu, P_\nu] = 0, \tag{8}$$

$$[P_\mu, M_{\nu\lambda}] = \delta_{\mu\nu} P_\lambda - \delta_{\mu\lambda} P_\nu, \tag{9}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho}, \tag{10}$$

$$[D, D] = 0, \tag{11}$$

$$[D, P_\mu] = -P_\mu, \tag{12}$$

$$[D, M_{\mu\nu}] = 0, \tag{13}$$

$$[D, K_\mu] = K_\mu, \tag{14}$$

$$[P_\mu, K_\nu] = 2(\delta_{\mu\nu} D - M_{\mu\nu}), \tag{15}$$

$$[M_{\mu\nu}, K_\lambda] = \delta_{\nu\lambda} K_\mu - \delta_{\lambda\mu} K_\nu, \tag{16}$$

$$[K_\mu, K_\nu] = 0. \tag{17}$$

Here we must note that relations (8)–(10) are Poincaré algebra. The Poincaré algebra is a sub-algebra of the conformal algebra.

In the next section we discuss about non-commutative space-time and obtain twisted space-time symmetries of relativistic particles action.

## 3 Non-commutative Symmetries

As we know, due to quantum gravity effects the space-time coordinates are non-commutative [9–11]. In non-commutative space-time, one can write,

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{k^2}\theta_{\mu\nu}^{(0)} + \frac{i}{k}\theta_{\mu\nu}^{(1)\rho}\hat{x}_\rho + i\theta_{\mu\nu}^{(2)\rho\tau}\hat{x}_\rho\hat{x}_\tau, \tag{18}$$

where the fundamental mass parameter $k$ has been introduced in order to exhibit the mass dimensions of respective terms and have the constant tensor $\theta_{\mu\nu}^{(0)}$, $\theta_{\mu\nu}^{(1)\rho}$, $\theta_{\mu\nu}^{(2)\rho\tau}$ as dimensionless. Twisted space-time symmetries of relativistic particles action has been studied in [6]. In here we have to study the invariance of local non-commutative action. Our goal is apply transformations (3)–(6) to relation (18) to find appropriate relations which leaves it invariant under that transformations. Equation (18) is invariant under the following transformation,

$$\hat{x}_\mu \to \hat{x}'_\mu = \hat{x}_\mu + \hat{a}_\mu, \tag{19}$$

if we have,

$$[\hat{a}_\mu,\hat{a}_\nu]=\frac{i}{k}\theta^{(1)\rho}_{\mu\nu}\hat{a}_\rho+i\theta^{(2)\rho\tau}_{\mu\nu}\hat{a}_\rho\hat{a}_\tau,$$
$$[\hat{x}_\mu,\hat{a}_\nu]=\frac{i}{2}\theta^{(2)\rho\tau}_{\mu\nu}(\hat{x}_\rho\hat{a}_\tau+\hat{x}_\tau\hat{a}_\rho). \tag{20}$$

Also (18) is invariant under the following transformation,

$$\hat{x}_\mu\to\hat{x}'_\mu=\hat{x}_\mu+\hat{\omega}^\mu_\nu\hat{x}^\nu, \tag{21}$$

if we have,

$$[\hat{x}_\mu,\hat{\omega}^\nu_\lambda\hat{x}^\lambda]=\frac{i}{2}\theta^{(2)\rho\tau}_{\mu\nu}(\hat{x}_\rho\hat{\omega}^\tau_\gamma\hat{x}^\gamma+\hat{\omega}^\rho_\gamma\hat{x}^\gamma\hat{x}_\tau),$$
$$[\hat{\omega}^\mu_\sigma\hat{x}^\sigma,\hat{\omega}^\nu_\lambda\hat{x}^\lambda]=\frac{i}{k}\theta^{(1)\rho}_{\mu\nu}\hat{\omega}^\rho_\gamma\hat{x}^\gamma+i\theta^{(2)\rho\tau}_{\mu\nu}\hat{\omega}^\rho_\gamma\hat{x}^\gamma\hat{\omega}^\tau_\varepsilon\hat{x}^\varepsilon. \tag{22}$$

Generally relation (18) is invariant under Poincaré transformation,

$$\hat{x}_\mu\to\hat{x}'_\mu=\hat{x}_\mu+\hat{a}_\mu+\hat{\omega}^\mu_\nu\hat{x}^\nu, \tag{23}$$

if we have the following relations

$$[\hat{a}_\mu,\hat{a}_\nu]=\frac{i}{k}\theta^{(1)\rho}_{\mu\nu}\hat{a}_\rho+i\theta^{(2)\rho\tau}_{\mu\nu}\hat{a}_\rho\hat{a}_\tau,$$
$$[\hat{x}_\mu,\hat{a}_\nu]=\frac{i}{2}\theta^{(2)\rho\tau}_{\mu\nu}(\hat{x}_\rho\hat{a}_\tau+\hat{x}_\tau\hat{a}_\rho),$$
$$[\hat{a}_\mu,\hat{\omega}^\nu_\lambda\hat{x}^\lambda]=\frac{i}{2}\theta^{(2)\rho\tau}_{\mu\nu}(\hat{a}_\rho\hat{\omega}^\tau_\gamma\hat{x}^\gamma+\hat{\omega}^\rho_\gamma\hat{x}^\gamma\hat{a}_\tau), \tag{24}$$
$$[\hat{x}_\mu,\hat{\omega}^\nu_\lambda\hat{x}^\lambda]=\frac{i}{2}\theta^{(2)\rho\tau}_{\mu\nu}(\hat{x}_\rho\hat{\omega}^\tau_\gamma\hat{x}^\gamma+\hat{\omega}^\rho_\gamma\hat{x}^\gamma\hat{x}_\tau),$$
$$[\hat{\omega}^\mu_\sigma\hat{x}^\sigma,\hat{\omega}^\nu_\lambda\hat{x}^\lambda]=\frac{i}{k}\theta^{(1)\rho}_{\mu\nu}\hat{\omega}^\rho_\gamma\hat{x}^\gamma+i\theta^{(2)\rho\tau}_{\mu\nu}\hat{\omega}^\rho_\gamma\hat{x}^\gamma\hat{\omega}^\tau_\varepsilon\hat{x}^\varepsilon.$$

But relation (18) is not invariant under scale and conformal transformation because the non-commutative version break the conformal and scale invariance. In that case we rewrite conformal algebra (8)–(17) in terms of new coordinates,

$$[\hat{P}_\mu,\hat{P}_\nu]=0, \tag{25}$$

$$[\hat{P}_\mu,\hat{M}_{\nu\lambda}]=\delta_{\mu\nu}\hat{P}_\lambda-\delta_{\mu\lambda}\hat{P}_\nu, \tag{26}$$

$$[\hat{M}_{\mu\nu},\hat{M}_{\lambda\rho}]=\delta_{\nu\lambda}\hat{M}_{\mu\rho}+\delta_{\mu\rho}\hat{M}_{\nu\lambda}-\delta_{\nu\rho}\hat{M}_{\mu\lambda}-\delta_{\mu\lambda}\hat{M}_{\nu\rho}, \tag{27}$$

$$[\hat{D},\hat{D}]=\frac{i}{k^2}\theta_{\mu\nu}(k\hat{x})\hat{P}_\mu\hat{P}_\nu, \tag{28}$$

$$[\hat{D},\hat{P}_\mu]=-\hat{P}_\mu, \tag{29}$$

$$[\hat{D},\hat{M}_{\mu\nu}]=0, \tag{30}$$

$$[\hat{D},\hat{K}_\mu]=\hat{K}_\mu+\frac{2i}{k^2}\theta_{\mu\nu}(k\hat{x})x^\mu\hat{P}_\mu\hat{P}_\nu, \tag{31}$$

$$[\hat{P}_\mu, \hat{K}_\nu] = 2(\delta_{\mu\nu}\hat{D} - \hat{M}_{\mu\nu}), \tag{32}$$

$$[\hat{M}_{\mu\nu}, \hat{K}_\lambda] = \delta_{\nu\lambda}\hat{K}_\mu - \delta_{\lambda\mu}\hat{K}_\nu, \tag{33}$$

$$[\hat{K}_\mu, \hat{K}_\nu] = \frac{4i}{k^2}\theta_{\mu\nu}(k\hat{x})\hat{x}^\mu\hat{x}^\nu\hat{P}_\mu\hat{P}_\nu, \tag{34}$$

where

$$\theta_{\mu\nu}(\hat{x}) = \theta^{(0)}_{\mu\nu} + \theta^{(1)\rho}_{\mu\nu}\hat{x}_\rho + \theta^{(2)\rho\tau}_{\mu\nu}\hat{x}_\rho\hat{x}_\tau. \tag{35}$$

We see that the Poincaré algebra (25)–(27) is same as old version in (8)–(10) but relations (28), (31) and (34) shows conformal symmetry break in non-commutative space-time. The algebraic relation between generators of scale and conformal transformations involves non-commutative parameter $\theta_{\mu\nu}$.

## 4 Conclusion

In this paper we consider symmetries of particles action. We study this symmetries in non-commutative space-time defined by (18). In here we obtained appropriate non-commutative relations which kept invariant the non-commutative Minkowski space-time. We rewrite conformal algebra in non-commutative space-time and shows that the non-commutative space-time break the conformal invariant as is expected, these lead us to have same symmetry for the massless and massive action.

## References


1. DeWitt, B.: In: Witten, L. (ed.) Gravitation (1962)
2. Bars, I.: Map of Witten's $*$ to Moyal's $*$. Phys. Lett. B **517**, 436 (2001)
3. Bars, I.: MSFT: Moyal star formulation of string field theory. arXiv:hep-th/0211238
4. Bars, I., Deliduman, C., Pasqua, A., Zumino, B.: Superstar in non-commutative super-space via covariant quantization of the super-particle. Phys. Rev. D **68**, 106006 (2003)
5. Chagas-Filho, W.: Relativistic particles and commutator algebras. arXiv:hep-th/0403136
6. Sadeghi, J., Pourhassan, B.: Relativistic particles and commutator algebras with twisted Poincaré transformation. Chaos Solitons Fractals **31**, 557 (2007)
7. Chan, C.-T., Lee, J.-C.: String symmetries and their high-energy limit. Phys. Lett. B **611**, 193–198 (2005). arXiv:hep-th/0312226
8. Zee, A.: Dark energy and the nature of the graviton. Phys. Lett. B **594**, 8–12 (2004). arXiv:hep-th/0309032
9. Doplicher, S., Fredenhagen, K., Roberts, J.E.: Phys. Lett. B **331**, 39 (1994)
10. Doplicher, S., Fredenhagen, K., Roberts, J.E.: Commun. Math. Phys. **172**, 187 (1995)
11. Kempf, A., Mangano, G.: Phys. Rev. D **55**, 7909 (1997)